# Department of Mathematics

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# Tutorial Sheet

**Topic: Linear Algebra**

**Vector space over real numbers/complex numbers, Subspaces, Linear span, Linearly independence, Dimensions and Basis, Linear transformations, Matrix of Linear transformation, Rank-Nullity theorem (Statement and their applications).**

**Vector space over real numbers/complex numbers**

1. **Problem:** Define vector space over real numbers/complex numbers.
   - Hint: A (real) vector space is a collection $V$ of vectors together with two binary operations, addition of vectors (+) and scalar multiplication of a vector by a real number (·), satisfying the following axioms:
     - Let $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be any vectors in $V$ and $\alpha$, $\alpha_1$,$\alpha_2$ be any (real number) scalars.
     - Note: The statement that + and · are binary operations means that $\mathbf{v}_1 + \mathbf{v}_2$ and $\alpha \cdot \mathbf{v}$ are always de ned; that is, they are both vectors in $V$.
     - [A1] Addition is commutative: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
     - [A2] Addition is associative: $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$.
     - [A3] There exists a zero vector $\mathbf{0}$, with $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
     - [A4] Every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$, with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
     - [M1] Scalar multiplication is consistent with regular multiplication: $\alpha_1 \cdot (\alpha_2 \cdot \mathbf{v}) = (\alpha_1\alpha_2) \cdot \mathbf{v}$.
     - [M2] Addition of scalars distributes: $(\alpha_1 + \alpha_2) \cdot \mathbf{v} = \alpha_1 \cdot \mathbf{v} + \alpha_2 \cdot \mathbf{v}$.
     - [M3] Addition of vectors distributes: $\alpha \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \alpha \cdot \mathbf{v}_1 + \alpha \cdot \mathbf{v}_2$.
     - [M4] The scalar 1 acts like the identity on vectors: $1 \cdot \mathbf{v} = \mathbf{v}$.
   - Remark: One may also consider vector spaces where the collection of scalars is something other than the real numbers: for example, there exists an equally important notion of a complex vector space, whose scalars are the complex numbers. (The axioms are the same, except we allow the scalars to be complex numbers.)

2. **Problem:** Prove the following:

   **a.** The vectors in $\mathbb{R}^n$ are a vector space, for any $n > 0$.

   **b.** The set of $m \times n$ matrices for any $m$ and any $n$, forms a vector space.

   **c.** The complex numbers (the numbers of the form $a + bi$ for real $a$ and $b$, and where $i^2 = -1$) are a vector space.

   **d.** The collection of all real-valued functions on any part of the real line is a vector space, where we de ne the sum of two functions as $(f + g)(x) = f(x) + g(x)$ for every $x$, and scalar multiplication as $(\alpha \cdot f)(x) = \alpha f(x)$.

**e.** The zero space with a single element **0**, with $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha \cdot \mathbf{0} = \mathbf{0}$ for every $\alpha$, is a vector space.

**Subspaces**

3. **Problem:** Define Subspace and Show that the set of diagonal 2 × 2 matrices is a subspace of the vector space of all 2 × 2 matrices.
4. **Problem:** Determine whether the set of vectors of the form ⟨t, t, t⟩ forms a subspace of $R^3$.
   Ans: Yes

   **Problem:** Determine whether the set of vectors of the form t, $t^2$ forms a subspace of $R^2$.
   Ans: No.
5. **Problem:** Determine whether the set of vectors of the form ⟨s, t, 0⟩ forms a subspace of $R^3$.
   Ans: Yes

6. **Problem:** The collection of solution vectors $\langle x_1, \cdots, x_n\rangle$ to any homogeneous system of linear equations forms a subspace of $R^n$.

   Ans: Yes

7. **Problem:** Determine whether the set of 2 × 2 matrices of determinant zero is a subspace of the space of all 2 × 2 matrices.
   Ans: No

8. **Problem:** Determine whether the set of 2 × 2 matrices of trace zero is a subspace of the space of all 2 × 2 matrices.

   Ans: Yes
9. **Problem:** Determine whether the set of vectors of the form $\langle x, y, z\rangle$ where $x, y, z \geq 0$ forms a subspace of $R^3$.
   Ans: No

10. **Problem:** Determine whether the set of vectors of the form $\langle x, y, z\rangle$ where $2x - y + z = 0$ forms a subspace of $R^3$.

    Ans: Yes

11. **Problem:** The collection of continuous functions on $[a, b]$ is a subspace of the space of all functions on $[a, b]$.

    Ans: Yes

12. **Problem:** The collection of $n$-times differentiable functions on $[a, b]$ is a subspace of the space of continuous functions on $[a, b]$, for any positive integer $n$.

    Ans: Yes

13. **Problem:** The collection of all polynomials is a vector space.
    Ans: Yes

14. **Problem:** The collection of solutions to the (homogeneous, linear) differential equation $y'' + 6y' + 5y = 0$ form a vector space.
    Ans: Yes

## Linear span

15. **Problem**: Determine whether the vectors ⟨2, 3, 3⟩ and ⟨4, −1, 3⟩ are in span (**v**, **w**), where **v** = ⟨1, −1, 2⟩ and **w** = ⟨2, 1, −1⟩.

    Ans: ⟨2, 3, 3⟩ is not in span and ⟨4, −1, 3⟩ is in span.

16. **Problem**: Show that the vectors ⟨1, 0, 0⟩, ⟨0, 1, 0⟩, and ⟨0, 0, 1⟩ span $R^3$.

17. **Problem**: Determine whether the vectors ⟨1, 2⟩, ⟨2, 4⟩, ⟨3, 1⟩ span $R^2$.

    Ans: Yes

18. **Problem**: Determine whether the vectors ⟨1, −1, 3⟩, ⟨2, 2, −1⟩, ⟨3, 4, 7⟩ span $R^3$.

    Ans: No

## Linearly independence

19. **Problem**: Determine whether the vectors ⟨1, 1, 0⟩, ⟨0, 2, 1⟩ in $R^3$ are linearly dependent or linearly independent.

    Ans: linearly independent

20. **Problem**: Determine whether the vectors ⟨1, 1, 0⟩, ⟨2, 2, 0⟩ in $R^3$ are linearly dependent or linearly independent.

    **Ans:** linearly dependent

21. **Problem**: Determine whether the vectors ⟨1, 0, 2, 2⟩, ⟨2, −2, 3, 0⟩, ⟨0, 3, 1, 3⟩, and ⟨0, 4, 1, 2⟩ in $R^4$ are linearly dependent or linearly independent.

    Ans: linearly dependent

22. **Problem**: Show that the functions $e^x$ and $e^{2x}$ are linearly independent in the vector space of all real-valued functions.

23. **Problem**: Show that the functions sin($x$) and cos($x$) are linearly independent using the Wronskian.

24. **Problem**: Determine whether the functions $1 + x$, $2 – x$, and $3 + 4x$ are linearly dependent or linearly independent.

    Ans: linearly dependent

## Dimensions and Basis

25. **Problem**: Show that the vectors ⟨1, 0, 0⟩, ⟨0, 1, 0⟩, ⟨0, 0, 1⟩ form a basis for $R^3$.

26. **Problem**: Show that the vectors ⟨1, 1, 1⟩, ⟨2, −1, 1⟩, ⟨1, 2, 1⟩ form a basis for $R^3$.

27. **Problem**: Show that the vectors ⟨1, 1, 0⟩ and ⟨1, 1, 1⟩ are not a basis for $R^3$.

28. **Problem**: Show that the vectors ⟨1, 0, 0⟩, ⟨0, 1, 0⟩, ⟨0, 0, 1⟩, ⟨1, 1, 1⟩ are not a basis for $R^3$.

29. **Problem**: Find a basis for the space $W$ of polynomials $p(x)$ of degree ≤ 3 such that $p(1) = 0$.

    Ans: $\left(x^3-1, x^2-1, x-1\right)$

30. **Problem:** Determine whether the vectors ⟨1, 2, 1⟩, ⟨2, −1, 2⟩, ⟨3, 3, 1⟩ form a basis of $R^3$.

    Ans: Yes

31. **Problem:** Show that the dimension of $R^n$ is $n$.

    Hint: Since the $n$ standard unit vectors form a basis.

## Linear transformations, Matrix of Linear transformation

32. **Problem:** If $V = W = \mathbb{R}^2$, show that the map $T$ which sends $\langle x, y\rangle$ to $\langle x, x + y\rangle$ is a linear transformation from $V$ to $W$.

33. **Problem:** If $V = M_{2\times2}(\mathbb{R})$ and $W = \mathbb{R}$, determine whether the trace map is a linear transformation from $V$ to $W$.

    Ans: Yes

34. **Problem:** If $V = M_{2\times2}(\mathbb{R})$ and $W = \mathbb{R}$, determine whether the determinant map is a linear transformation from $V$ to $W$.

    Ans: No

35. **Problem:** If $V$ is the vector space of all differentiable functions and $W$ is the vector space of all functions, determine whether the derivative map $D$ sending a function to its derivative is a linear transformation from $V$ to $W$.

    Ans: Yes

36. **Problem:** Show that if $V = W = \mathbb{R}^2$, then the map $T$ which sends $\langle x, y\rangle$ to $\langle ax + by, cx + dy\rangle$ for any $a, b, c, d$ is a linear transformation.
37. **Problem:** Show that if $V = \mathbb{R}^m$ (thought of as $m \times 1$ matrices) and $W = \mathbb{R}^n$ (thought of as $n \times 1$ matrices) and $A$ is any $n \times m$ matrix, then the map $T$ sending $\mathbf{v}$ to $A\mathbf{v}$ is a linear transformation.

## Rank-Nullity theorem (Statement and their applications)

38. **Problem:** If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is the linear transformation with $T(x, y) = (x + y, 0, 2x + 2y)$, then find a basis for the kernel and for the image of $T$.
    Ans: Basis for the kernel is given by the single vector (1, -1) and a basis for the image is given by the single vector (1,0,2).
39. **Problem:** If $T: M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ is the trace map, find the nullity and the rank of $T$ and verify the nullity-rank theorem.

    Ans: Nullity is 3 and rank is 1.

40. **Problem:** True or false

    *(a) There is a $3 \times 6$ matrix whose kernel is two-dimensional. Ans: False*

    *(a) If $A$ is a $4 \times 3$ matrix and the equation $A\vec{x} = \vec{0}$ has no nonzero solutions, then* $\operatorname{rank} A = 3$.

    *Ans: True*